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On some exact solutions in magnetohydrodynamics with astrophysical applications

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Some exact solutions of the steady magnetohydrodynamic equations for a perfectly conducting inviscid self-gravitating incompressible fluid are discussed. It is shown that there exist solutions for which the free surface of the liquid is that of a planetary ellipsoid and rotates with constant angular velocity about its axis. The stability of the equilibrium configuration is not investigated.

1. Introduction

The problem of mechanical equilibrium of a perfectly conducting incompressible fluid in the presence of a magnetic field has received considerable attention because of its importance in astrophysical applications. Ferraro (1954) considered the case of a uniform liquid star having uniform solid body rotation about its axis of symmetry in the presence of a certain poloidal magnetic field. He assumed that the gravitational energy of the star is much larger than its magnetic energy and obtained a first-order expansion for the surface of the star. Ferraro's work was extended by Roberts (1955) who obtained a series expansion for the surface of a body in the presence of Ferraro's field, but expressed some doubt about the convergence of the series. Lüst & Schlüter (1954) investigated the case where the electric current and magnetic field are parallel everywhere and thus the Lorentz force does not affect the fluid motion. Solutions for a large class of force free fields were produced by Chandrasekhar (1956*a*). Prendergast (1956) investigated the equilibrium of a self-gravitating liquid sphere in the presence of a poloidal magnetic field. The magnetic field is within the conductor and is zero on the surface of the liquid. For this case the Lorentz force is irrotational.

More recently Ranger (1970) produced some interesting generalizations of these solutions. He dealt with axisymmetric configurations where there is a finite, though not uniform, fluid motion inside a perfectly conducting liquid sphere in the presence of a magnetic field. Unfortunately not all solutions presented by Ranger make the pressure a constant on the fluid surface. Also for some of his solutions the electric current or the vorticity becomes infinite on the surface of the liquid. These solutions, of course, cannot have any significance for astrophysical applications. In this paper we examine these solutions and, since we refer them to astrophysical configurations, we also take account of the gravitational potential of the fluid. We show that when the fluid velocity has only an azimuthal component the appropriate free surface is that of a planetary ellipsoid.

The sphere is, of course, a special case of this configuration and corresponds to the case when the fluid velocity and magnetic field are zero on the surface of the fluid. We also discuss the case where the magnetic field has only an azimuthal component.

2. Equations of the problem

The steady-state equations for an inviscid perfectly conducting incompressible fluid of density ρ are

$$\rho \mathbf{v}.\nabla\mathbf{v} = -\nabla(p+\rho\Omega+\tfrac{1}{2}B^2)+\mathbf{B}.\nabla\mathbf{B}, \tag{1}$$

$$\nabla\times(\mathbf{v}\times\mathbf{B}) = 0, \tag{2}$$

$$\nabla.\mathbf{v} = 0, \quad \nabla.\mathbf{B} = 0, \tag{3}$$

where $\mathbf{v}$ is the velocity, $\mathbf{B}$ the magnetic field and Ω the gravitational potential per unit mass. Since we are going to apply our solution to astrophysical problems we must not ignore gravity. We make use of cylindrical polar co-ordinates (ϖ, ϕ, z) and consider the case where the velocity and magnetic field are symmetric about the z axis. The velocity and magnetic field may be expressed by

$$\mathbf{v} = \nabla\times\left(-\frac{\psi}{\varpi}\hat{\boldsymbol{\phi}}\right)+\frac{V}{\varpi}\hat{\boldsymbol{\phi}}, \tag{4}$$

$$\mathbf{B} = \nabla\times\left(-\frac{\chi}{\varpi}\hat{\boldsymbol{\phi}}\right)+\frac{U}{\varpi}\hat{\boldsymbol{\phi}}, \tag{5}$$

where ψ, χ, V and U are functions of ϖ and z.

Note that the flow field and magnetic field are interchangeable. [This becomes quite obvious if we replace $\mathbf{v}$ by $\mathbf{u}/\rho^{\frac{1}{2}}$ in (1), (2) and (3) and then take the curl of (1).] The interchange of $\mathbf{v}$ and $\mathbf{B}$, however, implies, in general, modification of p.

An obvious solution of (1), (2) and (3) is

$$\mathbf{B} = \rho^{\frac{1}{2}}\mathbf{v}, \quad p+\tfrac{1}{2}B^2 = \text{constant}.$$

The stability of this solution was discussed by Chandrasekhar (1956*b*).

Equations (1)–(3) possess exact solutions for several simple but interesting cases. These solutions have been discussed in some detail by Ranger (1970) for the case when the boundary of the fluid region is spherical. Below we re-examine these solutions and show that the appropriate boundary is that of an ellipsoid of revolution. The case of a spherical boundary is obtained from that of an ellipsoid when a certain parameter is set equal to zero.

Since the boundary of the fluid region is a streamline we must have ψ = constant, say $\psi = 0$, on the boundary. We also make the assumption that the exterior of the fluid region is non-conducting and therefore $\chi = 0$ on the boundary of the fluid region.

3. Flow field with only an azimuthal component

In this case $\psi = 0$ and the equations of the problem are satisfied (Ranger 1970) if we choose

$$V = \tilde{w}^2 f(\chi), \quad U = g(\chi),$$

where f and g are arbitrary functions and χ satisfies the equation

$$D^2\chi + \rho\tilde{w}^4 f(\chi) f'(\chi) + g(\chi) g'(\chi) = \tilde{w}^2 F(\chi). \tag{6}$$

Here a prime denotes differentiatiton, F is an arbitrary function and

$$D^2 = \frac{\partial^2}{\partial\tilde{w}^2} - \frac{1}{\tilde{w}}\frac{\partial}{\partial\tilde{w}} + \frac{\partial^2}{\partial z^2}.$$

The pressure p, obtained by integrating (1), is given by

$$p + \rho\Omega = \tfrac{1}{2}\rho\tilde{w}^2 f^2 - \int F(\chi)\, d\chi + \text{constant}. \tag{7}$$

Since on the free surface of the fluid region $\chi = 0$ and p must be zero we must have

$$\Omega_s = \tfrac{1}{2}\tilde{w}_s^2 f^2(0) + \text{constant}, \tag{8}$$

where a subscript s denotes a surface value of the variables. If $f(0) = 0$ then Ω_s is a constant and therefore the fluid region is a sphere. If $f(0) \neq 0$, then the only possible spheroidal body that satisfies (8) is the planetary ellipsoid, as was shown by Maclaurin. If G is the gravitational constant and e is the eccentricity of a meridian section of the ellipsoid then

$$f^2(0)/2\pi\rho G = (1-e^2)^{\frac{1}{2}} e^{-2}[(3-2e^2)\, e^{-1} \sin^{-1} e - 3(1-e^2)^{\frac{1}{2}}]. \tag{9}$$

The maximum value of the right-hand side of (9) is 0·2247, corresponding to $e = 0{\cdot}9299$. Thus if $f^2(0)/2\pi\rho G > 0{\cdot}2247$ there are no possible equilibrium configurations. For any smaller value of $f^2(0)/2\pi\rho G$ there are two possible equilibrium configurations the eccentricity being in one case less and in the other greater than 0·9299. Note that as $f(0) \to 0$ either $e \to 0$ and the ellipsoid tends to become a sphere or $e \to 1$ and the ellipsoid tends to become a disk. For details and further references on this see Lamb (1932) and Chandrasekhar (1969).

Further progress can only be made by assuming some forms for f, g and F. We set

$$ff' = K/\rho, \quad g = \alpha\chi \quad \text{and} \quad F = KA, \tag{10}$$

where K, α and A are constants and so

$$f^2 = 2K\chi/\rho + \omega_0^2, \tag{11}$$

where $\omega_0[=f(0)]$ is the constant angular velocity of the free surface where $\chi = 0$.

Since the fluid region is a planetary ellipsoid it is convenient to introduce the transformation

$$z = \kappa\cos\theta\sinh\eta = \kappa\mu\zeta, \quad \tilde{w} = \kappa\sin\theta\cosh\eta = \kappa(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}, \tag{12}$$

where $\mu = \cos\theta$ and $\zeta = \sinh\eta$. The surface of the ellipsoid is given by $\eta = \eta_0$ or $\zeta = \zeta_0$, where the eccentricity e, the semi-major axis a and semi-minor axis c are connected by

$$a = \kappa\zeta_0, \quad c = \kappa(\zeta_0^2+1)^{\frac{1}{2}}, \quad e = (\zeta_0^2+1)^{-\frac{1}{2}}.$$

On making use of (10) and (12), equation (6) becomes

$$\left\{\frac{1}{\zeta^2+\mu^2}\left[(\zeta^2+1)\frac{\partial^2}{\partial\zeta^2}+(1-\mu^2)\frac{\partial^2}{\partial\mu^2}\right]+\alpha^2\kappa^2\right\}\chi$$
$$= -K\kappa^6(1-\mu^2)(1+\zeta^2)[(1+\zeta^2)(1-\mu^2)-A_0], \quad (13)$$

where $A_0 = A/\kappa^2$.

The case $\alpha = 0$

In this case the magnetic field is poloidal. The solution for the homogeneous equation (13) is well known (Lamb 1932) and we construct a particular integral by setting

$$\chi = (1-\mu^2)f_1(\zeta)+(1-\mu^2)(5\mu^2-1)f_3(\zeta)+(1-\mu^2)(21\mu^4-14\mu^2+1)f_5(\zeta).$$

On substituting in (13) and equating coefficients of powers of μ, after some algebra, we find

$$(1+\zeta^2)f_1''-2f_1 = -(K\kappa^6/35)(1+\zeta^2)[4(1+\zeta^2)(1+7\zeta^2)-7A_0(1+5\zeta^2)], \quad (14)$$

$$(1+\zeta^2)f_3''-12f_3 = (K\kappa^6/15)(1+\zeta^2)[(1+\zeta^2)(3\zeta^2-1)+3A_0] \quad (15)$$

and
$$(1+\zeta^2)f_5''-30f_5 = (K\kappa^6/21)(1+\zeta^2)^2. \quad (16)$$

The solutions of (14), (15) and (16) corresponding to a finite velocity within the ellipsoid and $\chi(\zeta_0) = 0$ are

$$f_1 = (K\kappa^6/70)(\zeta_0^2-\zeta^2)(\zeta^2+1)(2\zeta^2+2\zeta_0^2+4-7A_0),$$

$$f_3 = -(K\kappa^6/90)\frac{(\zeta_0^2-\zeta^2)}{(5\zeta_0^2+1)}(\zeta^2+1)(5\zeta_0^2\zeta^2+\zeta^2+\zeta_0^2-3+9A_0)$$

and
$$f_5 = -(K\kappa^6/126)\frac{(\zeta_0^2-\zeta^2)(\zeta^2+1)}{(21\zeta_0^4+14\zeta_0^2+1)}(7\zeta_0^2\zeta^2+5\zeta^2+5\zeta_0^2+3).$$

It is obvious that this solution makes the magnetic field, electric current and fluid velocity finite within and on the ellipsoid. Also with the proper choice of K and A_0, for example, when $K^2\kappa^6/\rho$ and $K\kappa^6A_0/\rho$ are sufficiently small, f^2 (see (11)) is always positive within the ellipsoid and thus the vorticity is also finite everywhere in the fluid region. If in the above analysis we let $\omega_0 \to 0$, $\kappa \to 0$ and $\zeta \to \infty$ so that $\kappa\zeta = r$, $\kappa\zeta_0 = R$, where r is the distance from the origin, we have the case of a sphere. The radius of the sphere is, of course, R. Then if $K \neq 0$ as $r \to R$ the vorticity tends to infinity like $(R-r)^{-\frac{1}{2}}$.

The case $\alpha \neq 0$

In the spherical case ($\omega_0 = 0$) the solution of the equation corresponding to (13) involves the spherical Bessel functions $J_{\frac{3}{2}}$ and $J_{\frac{1}{2}}$ (Ranger 1970) and is straightforward. Indeed with the proper choice of A and α, one can make the vorticity finite on the surface of the sphere. For the ellipsoidal case considered here, however, the solution of (13) involves spheroidal wave functions of the first kind and is a little more involved.

For the complementary function of (13) we assume solutions of the form $X(\zeta)\,Y(\mu)$ and, after a little algebra, we obtain

$$(1-\mu^2)\,Y''+(\alpha^2\kappa^2\mu^2+b)\,Y=0, \tag{17}$$

$$(1+\zeta^2)\,X''+(\alpha^2\kappa^2\zeta^2-b)\,X=0, \tag{18}$$

where b is a separation constant.

The solutions of (17) and (18) that are finite within our ellipsoid are

$$(1-\mu^2)^{\frac{1}{2}}\,Ps_n^1(\mu,\,-\tfrac{1}{4}\alpha^2\kappa^2),$$

$$(1+\zeta^2)^{\frac{1}{2}}\,S_n^{1(1)}(-i\zeta,\,\tfrac{1}{4}\alpha^2\kappa^2),$$

where Ps_n^1 and $S_n^{1(1)}$ are spheroidal wave functions of the first kind. Here we are following the notation used in Erdélyi (1955) where information and further references on spheroidal wave functions may be found. We can now construct a solution of (13) satisfying the boundary condition $\chi(\zeta_0)=0$ as follows.

We multiply (13) throughout by $\zeta^2+\mu^2$ and express the right-hand side of the resulting equation in terms of Ps_{2n}^1. Thus our equation (13) becomes

$$\left[(\zeta^2+1)\frac{\partial^2}{\partial\zeta^2}+(1-\mu^2)\frac{\partial^2}{\partial\mu^2}+\alpha^2\kappa^2(\zeta^2+\mu^2)\right]\chi=\Sigma g_{2n}(\zeta)\,(1-\mu^2)^{\frac{1}{2}}\,Ps_{2n}^1(\mu,\,-\tfrac{1}{4}\alpha^2\kappa^2), \tag{19}$$

where g_{2n} is a cubic equation in ζ^2.

We now assume that

$$\chi=\Sigma G_{2n}(\zeta)\,(1-\mu^2)^{\frac{1}{2}}\,Ps_{2n}^1(\mu,\,-\tfrac{1}{4}\alpha^2\kappa^2).$$

If we substitute this expression in (19) and equate coefficients of Ps_{2n}^1 we obtain

$$(1+\zeta^2)\,G''_{2n}(\zeta)+(\alpha^2\kappa^2\zeta^2-b_{2n})\,G_{2n}(\zeta)=g_{2n}(\zeta). \tag{20}$$

The solution of this equation is

$$G_{2n}(\zeta)=[C(1+\zeta^2)^{\frac{1}{2}}\,S_{2n}^{1(1)}(-i\zeta,\,\tfrac{1}{4}\alpha^2\kappa^2)+H_{2n}(\zeta)],$$

where $H_{2n}(\zeta)$ is a particular integral and the constant C is adjusted so that $G_{2n}(\zeta_0)=0$.

It must be noted that, for all α, the poloidal magnetic field at the surface of the ellipsoid is tangential and not zero. Since it is assumed that the magnetic field external to the ellipsoid is zero there must exist a toroidal current sheet over the ellipsoidal surface. This exerts a magnetic pressure $\frac{1}{2}B_s^2$ normal to the surface of the ellipsoid in the outward direction. This pressure was ignored in the above analysis. The magnetic pressure was also ignored by Chandrasekhar (1956*c*) in a similar configuration. He assumed that the fluid is confined in its region. In view of (7) and the fact that $\chi=0$ at the boundary, we can neglect the magnetic pressure only when $B_s^2\ll 2\rho\Omega_s$. Our solution can therefore be interpreted as representing Maclaurin's spheroid in the presence of a small internal magnetic field which enables the interior of the spheroid to have a small additional angular velocity ω_1 which, in view of (11), is given by $\omega_1=K\chi/\rho\omega_0$.

We also note the existence of an exact solution in magnetohydrostatics in a perfectly conducting medium. This represents a poloidal magnetic field in the form of a spherical magnetic vortex embedded in an otherwise uniform external magnetic field, exactly like Hill's spherical vortex in a uniform stream, with the magnetic field just replacing the velocity field.

4. Flow field when the magnetic field has only a toroidal component

In this case $\chi = 0$, $\psi \neq 0$ and the roles of the magnetic field and velocity field in the equations already discussed are interchanged. We thus have

$$U = \tilde{\omega}^2 f(\psi), \quad V = g(\psi) \text{ and } D^2\psi + (\tilde{\omega}^4/\rho) f(\psi) f'(\psi) + g(\psi) g'(\psi) = \tilde{\omega}^2 F(\psi). \quad (21)$$

The pressure p, obtained by integrating (1), is given by

$$p + \rho\Omega + \tfrac{1}{2}\rho v^2 = -\tilde{\omega}^2 f^2 + \rho \int F(\psi)\, d\psi + \text{constant}. \quad (22)$$

Equation (22) is a little different from the corresponding expressions given by Ranger (1970). Thus the first term on the right-hand side of (22) is twice as large as the corresponding term of Ranger's equation (57). Also the right-hand side of (22) has a sign opposite to that of his equations (11) and (30).

Since $\psi = 0$ on the surface, if we exclude the case where electric currents are ejected from the surface of the fluid, though this case perhaps represents more realistic configurations for astrophysical applications, we must have $f(0) = 0$. The zero of $f(0)$, corresponding to $\psi = 0$ at the surface, must be at least of order one otherwise the electric current will tend to infinity as we approach the surface of the fluid. When these conditions are satisfied the magnetic pressure on the fluid surface is zero and the possible equilibrium configuration depends on v_s. If $v_s = 0$, the potential must be constant over the surface and thus the surface will be a sphere. Indeed, it was shown by Ranger (1970) that a solution where the fluid surface is spherical can be constructed if we set

$$f^2 = 2K\rho\psi, \quad g = \alpha\psi \quad \text{and} \quad F = KA, \quad (23)$$

where K, α and A are non-zero constants; K is arbitrary but α and A must be suitably chosen. This solution is in fact the solution referred to in the last section with the roles of **v** and **B** interchanged.

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